Human Crowds and Groups Interactions:
A Mean Field Games Approach

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Abstract— **This paper is devoted to some Mean Field Games (hereafter MFG) modeling of human crowds behavior. More precisely, we study 2-population dynamics (each of whom consisting of a continuum of individuals) with opposit interests. We focus on the crowd aversion case inside the group and also onto the other group (xenophobia in some cases). We write a *macro*-Nash problem between the two populations, then we give an existence and uniqueness result and characterize optimal points as MFG solutions. Finally we provide a simple gradient descent method to approximate the solutions and show some simulations.**

I. INTRODUCTION

MFG have been recently introduced by Lasry & Lions ([9], [10], [11]) and seem to be very useful to model big groups interactions with "intelligent" individuals (control aspects). To the best of our knowledge there exist very few papers on the topic (MFG and human crowds), e.g. the works of the author (see [6]) or Guéant (see [4]). The present paper is based on [7] and is mainly motivated by the macroscopic crowd motion models (see the recent work of Buttazzo, Jimenez & Oudet ([2]), Carlier & Salomon ([3]), Hughes ([5]), or Maury, Roudneff-Chupin & Santambrogio ([12]) for a gradient flow setting). Nevertheless, in our case, the MFG use corresponds to discrete-continuous fundaments. We adopt, as in [8], the optimal control point of view of MFG and take fully advantage of it to prove existence and to approximate solutions, in particular using the transformation originally due to Benamou & Brenier (see [1]). But the main point of this work is that we particularly focus on the modeling of interactions between two groups in some xenophobia situation.

The paper is organized as follows. In section II we briefly motivate the use of MFG for human crowds and present the problem we consider. Section III is devoted to the study of optimality conditions (that is in fact the link between MFG and optimal control) and existence. In the next part (section IV) we provide a numerical strategy to approximate the solutions, based on a gradient descent method. Finally, in section V we close with some simulations.

II. TWO-GROUP DYNAMICS: THE PROBLEM

A. MFG & human crowds

As mentioned in the introduction, it seems to be very natural to think of MFG to model human crowds. The MFG setting is well adapted for several reasons. First it is a micro-macro approach, being an approximation of N-player differential games when N tends to infinity (huge systems of individuals). Secondly it is useful to model interactions between "intelligent" individuals (or agents), that is: they are rational players with rational expectations. The next reason is that MFG describe non-cooperative equilibrium configurations (Nash point asymptotics). Note also that since MFG deal with atomized and anonymous agents (this last assumption is quite natural in human crowds), the mathematical framework (optimal control of Fokker-planck, MFG system) enables numerical simulations. To fix ideas, we now turn to recall briefly what are MFG in our crowd motion setting.

We work on the d-dimensional Torus $\Omega := \mathbb{T}^d$ to avoid boundary difficulties (note that they could be treated, as in [8] for instance). In the present work, we focus on the finite horizon framework. Define then the time-space domain, $Q = [0, T] \times \Omega$. To simplify, we consider in this paragraph a unique group (or population) composed with a continuum of individuals. It is fully characterized during the time period by the evolutive measure $(m_t)_{0 \leq t \leq T}$ (we will abusively refer to it as a density) with initial situation m_0 given. Then, the agent starting at $x \in \Omega$ at the beginning of the period evolves controlling the drift of the following stochastic process

$$dX_t^x = \alpha_t dt + \sigma dW_t \ , \ X_0^x = x,$$

where α_t is the control parameter, W_t a standard Brownian motion and σ the given noise.

Her individual problem is then to minimize (over a certain class of controls α) the quantity

$$\mathbb{E}\left[\int_0^T L(X_t^x, \alpha_t) + V[m_t](X_t^x)dt + \Psi(X_T^x)\right],$$

where the Lagrangian L stands for a control and position cost. In what follows we take the quadratic cost $L(x, \alpha) := \frac{|\alpha|^2}{2}$ (note that it does not depend on the position). Function Ψ is a final cost (incentive to reach a certain area) and V is a state cost (that is a criteria depending on the mean field created by the others, the density of agents). This last cost is the key point of the MFG modeling: individuals anticipate the crowd evolution $(m_t)_{0 \leq t \leq T}$ and take it into account in their optimization problem. It can be found in [10], [11], that the solution of the continuum of individual problems satisfies the so called *MFG system*

$$\partial_t m - \frac{\sigma^2}{2}\Delta m + \text{div}(m\partial_p H(x, \nabla v)) = 0$$
$$\partial_t v + \frac{\sigma^2}{2}\Delta v + H(x, \nabla v) = V[m],$$

with in the one hand initial and transversality conditions: $m|_{t=0} = m_0$ and $v|_{t=T} = \Psi$, in the other hand $\alpha := -\nabla v$ and H is the Legendre transform of L. It is well known (see [7], [8], [10]) that when V is the derivative (e.g. the Gâteaux derivative) of a potential Φ on bounded measures on Ω, i.e. $V = \Phi'$, then the critical points of the optimal control problem of Fokker-Planck

$$\begin{cases} \inf_\alpha \int_Q \frac{|\alpha|^2}{2}m + \int_0^T \Phi(m_t)dt + \int_\Omega \Psi m(T) \\ \partial_t m - \frac{\sigma^2}{2}\Delta m = -\text{div}(\alpha m), m(0) = m_0, \end{cases}$$
(1)

are solutions of the MFG system. Note also that it is a sufficient condition as soon as Φ is convex, which is the case in this paper. The convexity corresponds in fact to the crowd aversion (contrary to attraction situations, see for instance [7], [8]). In what follows, we call problem (1) a *global optimization problem* of a MFG.

B. Writing the Nash problem

Let us focus on the case where two populations interact inside Ω. We want to study equilibria between the two groups (typically Nash points as suggested by Lasry and Lions in [10]). Before giving the problem of group i, $i = 1, 2$, let us recall a classical notation. For any point $x = (x_1, x_2) \in \mathbb{R}^2$, and for a fixed coordinate i, we denote by x_{-i} the element of $\mathbb{R}$, $x_{-i} := \{x_1, x_2\} - \{x_i\}$. Formally, the global optimization problem (linked to a continuum of individual problems) of group

i, given the control and the mass evolution of the other group (i.e. $(\alpha_t^{-i}, m_t^{-i})$), reads as:

$$\inf_{\alpha^i} J_\lambda^i(\alpha)$$

where

$$J_\lambda^i(\alpha) := \int_Q \frac{|\alpha^i|^2}{2}m^i \quad + \quad \int_0^T \Phi_\lambda^i(m_t^1, m_t^2)$$
$$+ \int_\Omega \Psi^i m^i(T). \quad (2)$$

From now on, m^i depends on α^i, more precisely it is viewed as a bounded nonnegative measure (i.e. belonging to the set $\mathcal{M}_b(Q, \mathbb{R}_+)$) which is a weak solution of the Fokker-Planck equation:

$$\partial_t m^i - \frac{\sigma^2}{2}\Delta m^i = -\text{div}(\alpha^i m^i), m^i(0) = m_0^i.$$
(3)

We distinguish the populations by considering different initial densities $m_0^i(.)$ and different final incentive costs Ψ^i. However we study the simple case where the Brownian motion and the noise are similar for both groups. In the definition of the criterion (2), the coupling cost we consider is

$$\Phi_\lambda^i(m_t) := \int_\Omega (m_t^i)^2 + \lambda m_t^1 m_t^2,$$

for a nonnegative real constant λ. This models of a situation involving individual's aversion to the members of her own group and onto the other group, so that λ stands for a *"xenophobia"* parameter. To see that, note that the individual mean field criterion is:

$$V_i[m_t](x) = 2m_t^i(x) + \lambda m_t^{-i}(x).$$

In fact we consider that xenophobia is significant when the aversion λ to the other group is higher than the one to the own group (equal to 2). The Nash problem between the groups is then:

Find $\overline{\alpha} = (\overline{\alpha}^1, \overline{\alpha}^2)$ such that:

$(\mathcal{N}) \quad J_\lambda^i(\overline{\alpha}) = \inf_{\alpha^i \in \mathcal{M}_b(Q, \mathbb{R}^d)} J_\lambda^i(\alpha^i, \overline{\alpha}^{-i}),$

for $i = 1, 2$.

III. EXISTENCE & OPTIMALITY

A. Optimality conditions

In this part we present a characterization of the Nash equilibria. To do so, let us introduce the MFG system for two groups: for $i = 1, 2$,

$$\partial_t m^i - \frac{\sigma^2}{2}\Delta m^i + \text{div}(m^i \nabla v^i) = 0, \ m^i(0) = m_0^i,$$
(4)
$$\partial_t v^i + \frac{\sigma^2}{2}\Delta v^i + \frac{|\nabla v^i|^2}{2} = \Phi_\lambda^i(m)', \ v^i(T) = \Psi^i,$$
(5)

and the joint minimization problem (for the two groups)

$$(\mathcal{Q}) \inf_{\alpha=(\alpha^1,\alpha^2)} J_\lambda(\alpha) := J^1_{\lambda/2}(\alpha) + J^2_{\lambda/2}(\alpha),$$

under the constraints: m_i is a solution of (3), for $i = 1, 2$. Note that J_λ is convex if the xenophobia parameter is not too large, that is $\lambda \le 2$.

We are now in the position to give the optimality conditions.

Proposition 1: If $\lambda \le 2$ then the following assertions are equivalent:

1) $\overline{\alpha} \in \mathcal{M}_b(Q, \mathbb{R}^d)$ is a solution of $(\mathcal{N})$ and $\overline{m}$ satisfies (3) for $\alpha = \overline{\alpha}$,
2) $\overline{\alpha} \in \mathcal{M}_b(Q, \mathbb{R}^d)$ is a solution of $(\mathcal{Q})$ and $\overline{m}$ satisfies (3) for $\alpha = \overline{\alpha}$,
3) $(\overline{m}, \overline{v})$ is a solution of the MFG system (4)-(5), with $\alpha = \overline{\alpha} = \nabla\overline{v}$.

If $\lambda > 2$ then it is only necessary i.e. 2. $\Rightarrow$ 1.,3.

A proof of this statement is based on classical arguments of differential calculus and can be found in [7].

B. Existence

If there exists a solution of problem $(\mathcal{Q})$, then proposition 1 ensures the existence of a Nash point between the two groups. Before giving an existence result, we may reformulate the problem in a more rigorous way, following [1], [2], [8]. We adopt a vectorial point of view, and use the following notations:

- $\mathcal{M}_b(Q, \mathbb{R}^d)$ is the set of bounded d-vectorial measures on Ω.
- $m = (m^1, m^2) \in \mathcal{M}_b(Q, \mathbb{R}^2_+)$, and for all $x = (x^1, x^2) \in \mathbb{R}^2_{+*}$, $\frac{1}{x} := (\frac{1}{x^1}, \frac{1}{x^2})$,
- $q = (q^1, q^2) \in \mathcal{M}_b(Q, \mathbb{R}^{2d})$, and for all $y = (y^1, y^2) \in \mathbb{R}^d \times \mathbb{R}^d$, $|y|^2 := (|y^1|^2, |y^2|^2)$,
- $\mathcal{A} := \{(q, m) \in \mathcal{M}_b(Q, \mathbb{R}^{2d}) \times \mathcal{M}_b(Q, \mathbb{R}^2_+) : \int_Q (\partial_t u + \frac{\sigma^2}{2}\Delta u) dm^i + \int_Q \nabla u.dq^i = \int_\Omega \Psi^i m^i(T) - u_0 m^i_0 , \forall u \in C^\infty(Q)$, pour $i = 1, 2\}$.

Let us introduce the $(q = \alpha m, m)$ formulation, i.e. following [2], the functions

$$\varphi_1(a,b) := \begin{cases} \frac{|a|^2}{2b} & \text{if } (a,b) \in \mathbb{R}^{2d} \times \mathbb{R}^2_{+*} \\ +\infty & \text{else,} \end{cases}$$

$$\varphi_2(b) := \begin{cases} |b|^2 + \lambda b^1 b^2 & \text{if } b \in \mathbb{R}^2_{+*} \\ +\infty & \text{else,} \end{cases}$$

and

$$K(q,m) := K_1(q,m) + K_2(m),$$
$$K_1(q,m) := \int_Q \varphi_1(\frac{dq}{d\mathcal{L}^{d+1}}, \frac{dm}{d\mathcal{L}^{d+1}}) d\mathcal{L}^{d+1},$$
$$K_2(m) := \int_Q \varphi_2(\frac{dm}{d\mathcal{L}^{d+1}}) d\mathcal{L}^{d+1} + \int_\Omega \Psi.dm|_{t=T},$$

where $\mathcal{L}^{d+1}$ denotes the Lebesgue measure in $\mathbb{R}^{d+1}$. Then we can rewrite K in a simpler form:

$$K(q,m) = \begin{cases} J_\lambda(\alpha), & \text{if } q << m \text{ and } q = \alpha m \\ +\infty & \text{else.} \end{cases}$$

In this setting, a rigorous formulation of $(\mathcal{Q})$ is

$$\inf_{(q,m)\in\mathcal{A}} K(q,m), \tag{6}$$

and we can give the result as announced before.

Proposition 2: If $\lambda \le 2$ and $m^1_0, m^2_0 \in L^2(Q)$, then problem $(\mathcal{Q})$ possesses a solution (which is unique as soon as $\lambda < 2$). Moreover there exists a Nash point i.e. a solution of $(\mathcal{N})$.

Remark 1: Note that existence does not fail when adding a constraint of the type $m \le$ constant as in [12].

Given the reformulation (6), the proof is a simple adaptation of the one obtained by Buttazzo, Jimenez and Oudet in [2]. A complete proof of this proposition is provided in [7]. In the next section we deal with defining a numerical procedure to approximate the solution.

IV. NUMERICAL SETTING

In this part we introduce the discretization and a gradient descent method in order to approximate the solution(s) of problem $(\mathcal{N})$. More precisely, we distinguish the cases when the joint problem $(\mathcal{Q})$ is convex from when it is not. In the convex setting (i.e. when $\lambda \le 2$) we describe the gradient descent that we apply to the joint functional. The non-convex case $\lambda > 2$ (in which the xenophobia is significant) is more involved but interesting (we expect non-uniqueness). We then provide an alternating directions method taking advantage of the convexity of group i's problem, given group $(-i)$'s evolution.

Gradient First of all, let us write the gradient formula of the functional. We look at the reformulated problem given by (6). We slightly modify the point of view considering that the density m is an affine function of the momentum q. To fix ideas,

the joint problem reads as:

$$\inf_q F(q) := \sum_{i=1,2} \left(\int_Q \frac{|q_t^i|^2}{2m_t^i} + \Phi_{\lambda/2}^i(m_t) + \int_\Omega \Psi^i m_T^i \right),$$
(7)

where m^i, $i = 1, 2$ solves :

$$\partial_t m^i - \frac{\sigma^2}{2} \Delta m^i = -\text{div}(q^i) \, , \; m^i(0, .) = m_0^i(.).$$
(8)

Thanks to a classical differential calculus (see for instance [7]), it is easy to get the explicit formula of the gradient

$$\forall (q, m) \in \mathcal{A}, \forall w = (w^1, w^2) \in \mathcal{M}_b(\mathbb{R}^{2d}),$$

$$\nabla F(q).w = \left(\int_Q \left(\frac{q^i}{m^i} + \nabla \theta^i \right).dw^i \right)_{i=1,2}, \quad (9)$$

where θ^i satisfies, for $i = 1, 2$,

$$-\partial_t \theta^i - \frac{\sigma^2}{2} \Delta \theta^i = -\frac{|q^i|^2}{2(m^i)^2} + (2m^i + \lambda m^{-i}),$$
$$\theta^i|_{t=T} = \Psi^i. \quad (10)$$

Algorithm for the convex case Since problem (7) is convex when $\lambda \leq 2$, we decide to apply a gradient descent method. We focus on the 2D-case ($d = 2$) and take $\Omega = [0,1]^2$ with periodic boundary conditions.

Let M and N be two positive integers, we define the time and space steps by $dt = \frac{1}{N}$ and $dx = \frac{1}{M}$. For $(i, j, k) \in A := \{0, ..., N\} \times \{0, ..., M\}^2$, for a given function f defined on Q, $f_{j,k}^i$ denotes the numerical approximation of $f(idt, jdx, kdy)$. Equations (8) and (10) are iteratively solved by using finite differences, after initializations ($m_{j,k}^0 = m_0(jdx, kdy)$ and $\theta_{j,k}^N = \Psi(jdx, kdy)$), using the following approximations

$$\partial_t f(idt, jdx, kdy) = \frac{f_{j,k}^{i+1} - f_{j,k}^i}{dt},$$

$$\Delta f(idt, jdx, kdy) = \frac{f_{j+1,k}^i - 2f_{j,k}^i + f_{j-1,k}^i}{(dx)^2}$$
$$+ \frac{f_{j,k+1}^i - 2f_{j,k}^i + f_{j,k-1}^i}{(dy)^2}.$$

At step n, let $f^{(n)} := \left(f_{j,k}^{i,(n)} \right)_{(i,j,k) \in A}$. Then the gradient descent method (hereafter GDM) is the following:

1) *Initialization:*
   Choose $q^{(0)}$ then compute $m^{(0)}$ by solving (8) with the finite difference scheme.
2) *Step n:*

- Compute $\theta^{(n)}$ by solving numerically (10) with $q^{(n-1)}$ and $m^{(n-1)}$, then compute the discretized gradient $\nabla F(q^{(n-1)})$ (formula (9)), using $\theta^{(n)}$.
- Compute the descent: $q^{(n)} = q^{(n-1)} - \rho_n \nabla F(q^{(n-1)})$.
- If $||q^{(n)} - q^{(n-1)}|| <$Tol1, then stop the algorithm (Tol1 is a tolerance threshold defined by the user).
  Else, $n = n + 1$.

Note that ρ_n above is the descent step size, it is chosen optimal, i.e. minimizing the following: $\rho \in [0, 1] \rightarrow F(q^{(n-1)} - \rho \nabla F(q^{(n-1)}))$.

Alternating directions method for the non-convex problem The case where aversion to the other group is significant ($\lambda > 2$), for which we have less theoretical results, also seems interesting. One of the main goal of the present work is to obtain numerical simulations in this situation. Consequently, it is convenient to describe the way we try to approximate the Nash points between the groups when $\lambda > 2$. To do so we choose an alternating directions method, provided the convexity of both group i's problem, given group $(-i)$'s evolution:

$$\inf_{q^i} F^i(q) := \int_0^T \int_\Omega \frac{|q_t^i|^2}{m_t^i} + \Phi_\lambda^i(m_t)dt + \int_\Omega \Psi^i m_T^i,$$

where m^i solves (8) for q^i, $i = 1, 2$. One can easily get the formula of the gradient of F^i looking at the joint case (9)-(10).

In what follows, by writing that we compute q we also mean that we compute the corresponding m solution of the discretized versions of the Fokker-Planck equations (8), for $i = 1, 2$.

The strategy to approximate the Nash points is to apply GDM successively to each group. Note that above, the upper index refers to the group number and the lower one to the iteration.

1) *Initialization:*
   Choose q_0^1 then compute q_1^2 with GDM and q_0^1.
2) *Step $k \geqslant 1$:*
   We know q_k^2.

   - Compute q_k^1 then q_{k+1}^2 by using GDM (with, respectively, q_k^2 and q_k^1).
   - If $||q_k^i - q_{k-1}^i|| <$Tol2 for $i = 1, 2$, then stop the procedure.
     Else, $k = k + 1$.

V. SIMULATIONS

The GDM shows good convergence results when the initials densities of individuals are significantly

positive (i.e. $m_0^i > \text{constant} > 0$). This last section is devoted to the exhibition of some very first tests. In the next simulations we take $T = 1$ and $\frac{\sigma^2}{2} = 0.01$.

A. Test 1: crowd aversion in a single group

In the first example we focus on a case involving only one population ($m_0^2 = 0$), i.e. a similar framework as the one studied by Buttazzo, Jimenez and Oudet in [2]. Fig. 1 shows the initial density of agents (centralized around the point $(0.1, 0.1)$) and the final cost, modeling a strong incentive for individuals to be in some neighborhood of $(0.5, 0.8)$ and $(0.8, 0.5)$ at instant T.

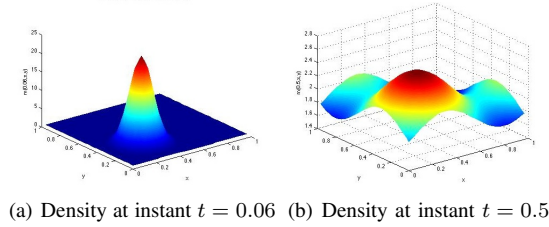(a) Initial density   (b) Final cost Ψ

Fig. 1. Data

(a) Density at instant $t = 0.06$ (b) Density at instant $t = 0.5$
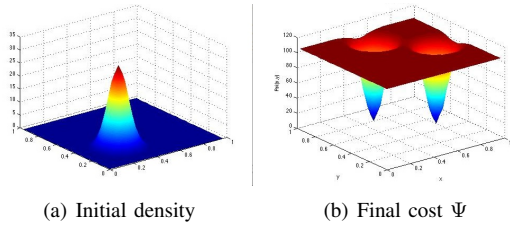
Fig. 2. Spreading over during the first half

(a) Density at instant $t = 0.9$ (b) Density at instant $T = 1$
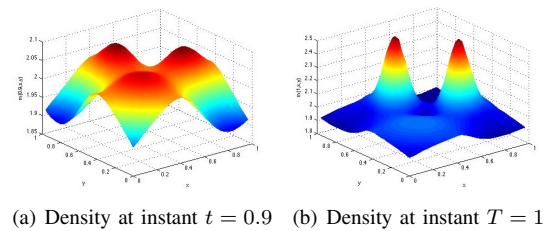
Fig. 3. Splitting and centralization during the second half

Fig. 2 and Fig. 3 present the mass evolution at some chosen instants in $[0, T]$. More precisely, we may observe on Fig. 2 a first step corresponding to a spreading over of m (explained by the aversion term and the diffusion parameter). Note that the running time of dispersion is greater than one half. We then observe in Fig. 3 a split inside the population so that individuals can converge to the two attractive areas. Finally, the discrete energy
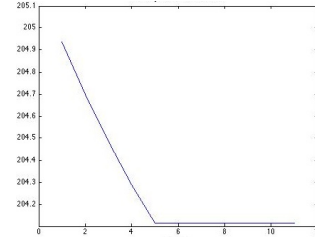

Fig. 4. Value of F for each iteration

seems to reach quickly the minimum (5 iterations), see Fig. 4.

B. Test 2: groups interactions

Let us now look at the more interesting case with two populations. Recall that we look for Nash equilibria between two groups whose global optimization problem is (6). We use the procedure detailed before (starting with group 2). In order to emphasize the xenophobia behavior we choose $\lambda = 20$ in the definition $\Phi_\lambda^i(m^1, m^2) = \int_\Omega (m^i)^2 + \lambda m^1 m^2$. We consider a symmetric configuration and represent the graphs of the initial densities (m_0^i, $i = 1, 2$) and final costs (Ψ^i, $i = 1, 2$) in Fig. 5. Group 1 is initially centralized around $(0.35, 0.5)$, group 2 around $(0.5, 0.35)$. Concerning the final costs the situation is still symmetric since they model incentives to reach positions in the neighborhood of (respectively for group 1 and 2) $(0.65, 0.5)$ and $(0.65, 0.5)$.

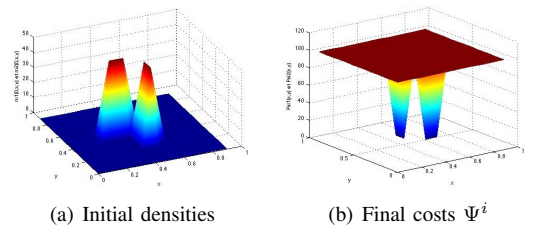With such a situation we are interested in crossing phenomenon.

(a) Initial densities   (b) Final costs Ψ^i

Fig. 5. Data

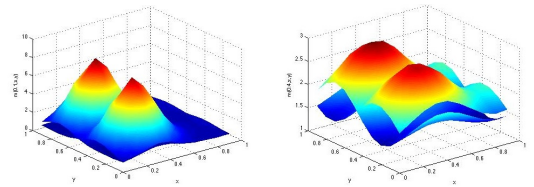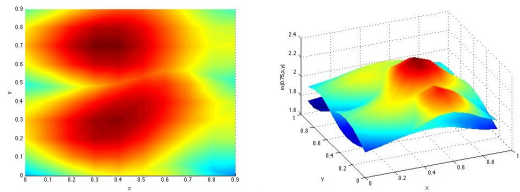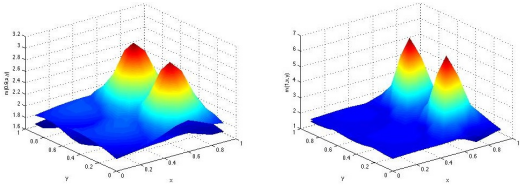(a) Densities at instant $t = 0.1$ (b) Densities at instant $t = 0.4$

Fig. 6. Spreading over of m^1 and m^2

(a) Densities contour lines at instant $t = 0.65$

(b) Densities at instant $t = 0.75$

Fig. 7. Group 2 go through the middle of the domain as some agents of group 1, nevertheless most of group 2 individuals transit by the border (periodic conditions)

(a) Densities at instant $t = 0.9$

(b) Densities at instant T=1

Fig. 8. End of the period evolution

The graphs of both densities are depicted in Fig. 6. We can notice the same spreading over we observed in Test 1. However, the most interesting evolution period is described in Fig. 7. Indeed, we can see that group 1 gives the priority to group 2 to go to its attractive area passing through the center of the domain (the shortest road for the euclidian metric). Some of the individuals of group 1 wait, some others go through the border (periodic conditions), and the lasts go through the center (the most congested area). Anyway we note that group 2 reaches quicker than group 2 its goal. Looking at Fig. 8, we can check that both group are finally centralized around the points $(0.65, 0.5)$ and $(0.5, 0.65)$.

The last remark is that we obtain the opposite (or symmetric) situation when starting to optimize on group 1. Then, the symmetry break seems to be a consequence of this starting choice.

VI. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

In this work we provide a macroscopic model for human crowds and groups interactions for intelligent individuals. It is a MFG model also inspired by [2]. We take advantage of the optimal control point of view of MFG in order to prove existence and to develop a numerical approach which consists of a simple gradient descent method. We also test the algorithm in a case involving xenophobia between two groups.

B. Future works

To the best of our knowledge this paper is one of the first works exploring both theoretically and numerically a MFG approach to model groups interactions and human crowds, so that many things remain to be done. For instance we think of: proving the gradient method convergence, testing its robustness when $\sigma \to 0$, or taking into account congestion effects (i.e. considering moving cost of the type $L = |\alpha|^2 m^b$ with $b > 0$).

VII. ACKNOWLEDGMENTS

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